Ambedkar University Delhi

Exam I: PhD Programme in Mathematics (PAPER A)

Both Part A and Part B are compulsory. Part A consists of multiple-choice questions which need to be completed in a maximum of 1 hour. Total duration of the exam is **maximum 3 hours.**

PART A: Section I

Tick the single correct answer in the following questions. Use only pen to tick correct answers. Overwriting in the answer to an objective question will mean zero marks for that question. Each question carries 3 marks.

1. Let G be a non-abelian group of order p^3 where p is prime.
   (a) G has a non-central element whose centraliser has order p^3.
   (b) G has a non-central element whose centraliser has order p^2.
   (c) G has a non-central element whose centraliser has order p.
   (d) None of the above.

2. Consider polynomials $p(x) = (x-1)^{2a} - x^{2a} + 2x - 1$ and $q(x) = 2x^3 - 3x^2 + x$. Then
   (a) $p(x)$ is not divisible by $q(x)$ for any positive integer a.
   (b) $p(x)$ is divisible by $q(x)$ for all positive integer $a \geq 2$.
   (c) $p(x)$ is divisible by $q(x)$ only for $a = 2$.
   (d) None of above.

3. Value of the integral
$$\int\limits_{|z|=1} \frac{f'(z)}{f(z)} dz =$$
   where $f(z) = (z^2+4)/(z-3/2)$
   (a) 0 .
   (b) 1 .
   (c) 2.
   (d) -1.

4. The nth divided difference of $f(x) = x^n$ at $x = x_0, x_1, ...x_n$ is
   (a) 0 .
   (b) n! .
   (c) 1.
   (d) $(-1)^{n-1}$.

5. The flux of the vector field $\vec{F} = z\hat{i} + y\hat{j} + x\hat{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$ is given by.

(a) $2\pi/3$
(b) $4\pi/3$.
(c) $\pi/3$
(d) π

6. The complete integral of $p^3 + q^3 - 3pqz = 0$. where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. is:

(a) $(1 + a^3)logz = 3a(x + ay) + b$.
(b) $(1 + a^2)logz = 3a(ax + y) + b$.
(c) $(1 + a^3)logz = 2a(x + by) + a$.
(d) $(1 + a)logz = a(x + ay) + b$.

7. Solution of the differential equation $x\frac{dy}{dx} = y(lny - lnx + 1)$ is:

(a) $y = xe^{cx}$.
(b) $y + xe^{cx} = 0$.
(c) $y + ce^{x} = 0$.
(d) $x = e^{cx}$
where c is an arbitrary constant.

8. The highest power of 5 dividing 2000! is

(a) 496.
(b) 497.
(c) 498.
(d) 499.

PART A: Section II

Question 1 and Question 2 are compulsory. Choose any two from Question 3, Question 4 , Question 5 and Question 6. Tick the all correct answer in the following questions. Marks will be awarded only on identification of "ALL" correct options. Use only pen to tick correct answers. Overwriting in the answer to an objective question will mean zero marks for that question. Each question carries 4 marks.

1. Which of the choices given below the statements consists of all correct answers.

(a) A cyclic group can only have a cyclic automorphism group.
(b) A cyclic group can only have an abelian automorphism group.
(c) A non-abelian group with a trivial centre can have an abelian automorphism group.
(d) An abelian group can have a non-abelian automorphism group.

2. Let M be a subspace of a Hilbert space H. Then

(a) M is complete if M is closed in H.
(b) M is closed if M is complete.

(c) If M is finite dimentional then M is complete.

(d) If H is seperable then M may not be seperable.

3. Let $f(z) = (z^3+1)\cos z^3$ for $z \in \mathbb{C}$. Let $f(z) = u(x,y) + iv(x,y)$, where u and v are real and imaginary part of $f(z)$. Then

   (a) u is continuous but need not be differentiable.

   (b) u is infinite times differentiable.

   (c) u is bounded.

   (d) $f(z)$ can be expressed as a Taylor's series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$.

4. Choose the correct statements.

   (a) In $\mathbb{Z}[i\sqrt{3}]$ the element $1 + i\sqrt{3}$ is reducible but not a prime.

   (b) The ring $\mathbb{Z}[i\sqrt{3}]$ is factorization domain but not unique factorization domain.

   (c) The ring of all complex entire functions is not a factorization domain.

   (d) The rings $\mathbb{Z}$, $\mathbb{Z}[x]$ are euclidean domain but $\mathbb{Z}[[x]]$ is not.

5. The function $f(x,y) = xy$

   (a) satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$ and $c \leq x \leq d$.

   (b) does not satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$ and $c \leq x \leq d$.

   (c) satisfies a Lipschitz condition on any strip $a \leq x \leq b$ and $-\infty < y < \infty$.

   (d) does not satisfies a Lipschitz condition on any strip $a \leq x \leq b$ and $-\infty < y < \infty$.

6. Which of the following statements are correct:

   (a) 3 is a quadratic residue of 23.

   (b) If $\tau(n)$ denotes the number of divisors of n, then $\tau(360) = 20$.

   (c) Number of primitive roots of 19 is 6.

   (d) The Diophantine equation $6x + 21y = 23$ is solvable.

(Part B)

Each question carries 15 marks. Attempt any two questions.

1. Evaluate using contour integration

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x} dx$$

2. By using the fact that $\mathbb{Z}[x]$ is unique factorization domain show that one 2-sided die labelled with 1 and 4 and another 18-sided die labelled with $1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 8$ yield the same probabilities as an ordinary pair of cubes labelled 1 through 6.

3. (a) Show that

$$\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}$$

for $x \in \mathbb{R}$ and $n \geq 0$.
(b) Consider the polynomials $P_n(f.x)$ defined by

$$P_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

Show that $P_n(f;x) \to f$ uniformly on $[0,1]$.
(c) Does there exist a sequence of polynomials converging uniformly on $\mathbb{R}$ to $f(x) = \sin x$?

4. Determine the solution of the following initial boundary-value problem
$u_{tt} - 4u_{xx} = 0,\ 0 < x < \infty,\ t > 0.$
$u(x,0) = x^2/8\ ,\ 0 \leq x < \infty,$
$u_t(x,0) = x,\ 0 \leq x < \infty,$
$u(0,t) = t^2,\ t \geqslant 0.$